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Exponential Stability Criterion for Uncertain Retarded Systems with Multiple Time-Varying Delays

Yeong-Jeu Sun, Jer-Guang Hsieh, and Yih-Chang Hsieh

Department of Electrical Engineering, National Sun Yat-Sen University, Kaohsiung, Taiwan, Republic of China, 80424

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In this paper, an easy-to-check exponential stability criterion for a class of uncertain retarded systems with multiple time-varying delays is proposed. An estimate of the convergence rate is also derived. Furthermore, a numerical example is given to illustrate our main results.

1. INTRODUCTION

In recent years, the stability problem of retarded systems has been widely investigated; see, for example, [1–4] and [6–17]. This is due to theoretical interests as well as to a powerful tool for practical system analysis and control design, since delays are often encountered in various engineering systems, such as the turbojet engine, the ship stabilization, chemical engineering systems, the microwave oscillator, the rolling mill, systems with lossless transmission lines, the manual control, and the nuclear reactor. Frequently, it is a source of the generation of oscillation and a source of instability in many control systems. However, in many practical retarded systems, the time delays appearing in the systems are time-varying or are only known to be bounded by some constants. Typical retarded systems with multiple time-varying delays include microwave oscillator, turbojet engine, inferred grinding model, control of epidemics, and population dynamics model. Consequently, the problem of stability analysis of retarded systems with time-varying delays has been a main concern of researchers [8]. Depending on whether the stability criterion itself contains the delay argument as a parameter, the stability criteria for retarded systems can be classified into two categories, namely delay-inde-

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pendent criteria and delay-dependent criteria. Generally speaking, the former ones are more conservative than the latter ones, which, in general, may not be easily verified. There have been a number of interesting developments in searching the stability criteria for retarded systems, with or without uncertainties, but most have been restricted either to searching the delay-dependent criteria for the asymptotic stability of retarded systems with commensurate delays or to searching the delay-independent criteria for the asymptotic stability of retarded systems with constant time delays; see, for example, [2, 7, 9, 10, 13, 14, 16]. Only a few developments have dealt with searching the delay-independent criteria for the asymptotic stability of retarded systems with time-varying delays cases; see, for example, [3]. It is the purpose of this paper to search a delay-dependent criterion under which the exponential stability of a class of uncertain retarded systems with multiple time-varying delays can be guaranteed. Moreover, the result can be applied to the stabilizability problem of a class of retarded systems with multiple time-varying delays.

This paper is organized as follows. In Section 2, a criterion is proposed to guarantee the exponential stability for a class of uncertain retarded systems with multiple time-varying delays. An estimate of the convergence rate is also given. In Section 3, an example is provided to illustrate our main results.

2. MAIN RESULTS

For convenience, we define some notations that will be used throughout this paper as follows:

$$
\begin{aligned}
C^{m\times n} &:= \text{the set of all complex } m \text{ by } n \text{ matrices},\\
\underline{C}(t_1) &:= \{\phi\colon [t_1 - H, t_1] \to \Re^n : \phi \text{ is continuous}\},\\
\overline{C}(t_1) &:= \{\phi\colon [t_1 - 2H, t_1] \to \Re^n : \phi \text{ is continuous}\},\\
|a| &:= \text{the absolute value of the real number } a,\\
\mathrm{Re}[\lambda] &:= \text{the real part of the complex number } \lambda,\\
I &:= \text{the unit matrix},\\
A^* &:= \text{the conjugate transpose of the matrix } A,\\
\lambda_i(A) &:= \text{the } i\text{th eigenvalue of the matrix } A,\\
\mathrm{diag}(\lambda_i(A)) &:= \text{the diagonal matrix with diagonal elements } \lambda_i(A),\\
\|A\| &:= \text{the induced Euclidean norm of the matrix } A,\\
\mu(A) &:= \text{the matrix measure of } A;\ \mu(A) = \tfrac{1}{2}\lambda_{\max}[A^* + A],\\
\kappa(P) &:= \text{the condition number of the matrix } P;\ \kappa(P) = \|P\|\cdot\|P^{-1}\|,\\
\underline{m} &:= \{1, 2, \ldots, m\},\\
\overline{m} &:= \{0, 1, \ldots, m\}.
\end{aligned}
$$

Consider the following uncertain retarded system with multiple time-varying delays:

$$\begin{aligned}\dot{x}(t) &= F(t, x_t)\\ &= A_0 x(t) + \sum_{i=1}^{p} A_i x(t - h_i(t))\\ &\qquad + f\big(t, x(t), x(t - h_1(t)), \ldots, x(t - h_p(t))\big), \qquad t \geq 0, \quad (1a)\\ x(t) &= \theta(t), \qquad t \in [-H, 0]; \quad (1b)\end{aligned}$$

where $x \in \Re^n$ is the state vector, x_t is the segment of $x(s)$ for $t - H \leq s \leq t$ with $H \geq 0$, $x_t(s) := x(t+s)$, $\forall s \in [-H, 0]$, with $\|x_t\|_s := \sup_{-H \leq r \leq 0} \|x(t+r)\|$, $A_i \in \Re^{n \times n}$, $\forall i \in \bar{p}$, f, the uncertain term, is a smooth vector-valued function with $f(t, 0, \ldots, 0) = 0$, $h_i(t)'s$, $\forall i \in \underline{p}$, are arbitrary delay arguments with $0 \leq h_i(t) \leq H$ for some constant $\bar{H}$, and $\theta(t)$ is a given continuous vector-valued initial function.

Definition 1 [3, 15]. The system (1) is said to have a stability degree α (or to be exponentially stable), with $\alpha > 0$, if the state of (1) can be written as $x(t) = [\exp(-\alpha t)] z(t)$ and the system governing the state $z(t)$ is globally asymptotically stable. In this case, the parameter α is called the convergence rate.

The following assumption is made on the system (1) throughout this paper.

(A1) There exist non-negative constants $a_i' s$, $\forall i \in \bar{p}$, such that, for all arguments,

$$\|f(t, z_0, z_1, \ldots, z_p)\| \leq \sum_{i=0}^{p} a_i \cdot \|z_i\|.$$

Before presenting our main results, let us introduce two lemmas which will be used in the proof of our main theorems.

Lemma 1 [6, Theorem 4.2]. *Suppose that $F : \Re \times C(0) \to \Re^n$ is continuous, locally Lipschitz in x_i, and takes $\Re \times$ (bounded sets of $C(0)$) into bounded sets of $\Re^n$; $u, v, w : \Re^+ \to \Re^+$ are continuous, non-decreasing functions; $u(s)$, $v(s)$, $w(s)$ are positive for $s > 0$, $u(0) = v(0) = 0$; and*

$u(s) \to \infty$ as $s \to \infty$. If there exist a continuous function $V : \Re \times \Re^n \to \Re$ and a continuous non-decreasing function $p(s) > s$ for $s > 0$ such that

$$u(\|x\|) \leq V(t, x) \leq v(\|x\|), \qquad \forall t \in \Re, x \in \Re^n,$$

and

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|) \qquad \text{if } V(t + r, x(t + r)) < p(V(t, x(t))), \quad \forall -H \leq r \leq 0,$$

where $\dot{V}$ is the derivative of V along the solutions of the system (1), *then the system* (1) *is globally asymptotically stable.*

LEMMA 2 [5]. *Let A, $B \in C^{n \times n}$, then we have*

(i) $\mu(A + B) \leq \mu(A) + \mu(B)$;

(ii) $\mu(A + \alpha I) = \mu(A) + \alpha, \forall \alpha \in \Re$.

LEMMA 3. *Let $A \in C^{n \times n}$, then for any given constant $a > 0$, there exists a nonsingular matrix T such that*

$$\mu(TAT^{-1}) \leq \max_i \operatorname{Re}[\lambda_i(A)] + a.$$

Proof. Let J be the Jordan form of A, then $J = D + N$, where D is diagonal with eigenvalues of A on its main diagonal and all elements of N are either 0 or 1 with all nonzero elements located on the diagonal above the main diagonal. Let M be the modal matrix of the matrix A, i.e., $J = M^{-1}AM$. For any $\delta > 0$, define

$$P = \operatorname{diag}[1 \ \ \delta \ \ \delta^2 \ \cdots \ \delta^{n-1}] \qquad \text{and} \qquad T = P^{-1}M^{-1},$$

then $TAT^{-1} = P^{-1}JP = D + \delta N$. Note that $\mu(N) \geq 0$. It is easy to see that

$$\begin{aligned} -\mu(TAT^{-1}) &= -\mu(P^{-1}JP) \\ &= -\mu(D + \delta N) \geq -\mu(D) - \delta\mu(N) \\ &= -\max_i \operatorname{Re}[\lambda_i(A)] - \delta\mu(N), \end{aligned}$$

in view of Lemma 2 (i). This implies that

$$\mu(TAT^{-1}) \leq \max_i \operatorname{Re}[\lambda_i(A)] + \delta\mu(N).$$

If $\mu(N) = 0$, which corresponds to the case when A is diagonalizable, then

$$\begin{aligned}\mu(TAT^{-1}) &\leq \max_i \operatorname{Re}[\lambda_i(A)] + \delta\mu(N) \\ &< \max_i \operatorname{Re}[\lambda_i(A)] + a, \qquad \text{for any } \delta > 0.\end{aligned}$$

If $\mu(N) \neq 0$, we may choose $\delta = a/\mu(N) > 0$ to obtain

$$\mu(TAT^{-1}) \leq \max_i \operatorname{Re}[\lambda_i(A)] + \delta\mu(N) = \max_i \operatorname{Re}[\lambda_i(A)] + a.$$

This completes our proof. ∎

Now we present our main result for the exponential stability of the system (1).

THEOREM 1. *The system* (1) *satisfying* $(A1)$ *is exponentially stable provided that there exist a nonsingular matrix T and a set $L \subseteq \underline{p}$ such that*

$$\begin{aligned}\mu(TAT^{-1}) &+ H \cdot \left(\sum_{i \in L} \|A_i\|\right) \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_i\right) \\ &+ \left(\sum_{i \in \underline{p} \setminus L} \|TA_iT^{-1}\|\right) + \kappa(T) \sum_{i=0}^{p} a_i < 0, \end{aligned} \tag{2}$$

where

$$A = A_0 + \sum_{i \in L} A_i. \tag{3}$$

In this case, the guaranteed convergence rate is given by $\alpha := \bar{\alpha} - \varepsilon$, where $\bar{\alpha}$ is the unique positive zero of the function

$$\begin{aligned} D := \mu(TAT^{-1}) &+ x \\ &+ [\exp(2Hx)] \cdot H \cdot \left(\sum_{i \in L} \|A_i\|\right) \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_i\right) \\ &+ [\exp(Hx)] \cdot \left(\sum_{i \in \underline{p} \setminus L} \|TA_iT^{-1}\|\right) + [\exp(Hx)] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i, \\ &\qquad x \geq 0, \end{aligned} \tag{4}$$

and ε is any arbitrarily small positive number with $\varepsilon < \bar{\alpha}$.

Proof. Without loss of generality, we may let $L = \{1, 2, \ldots, m\} \subseteq \underline{p}$.

Then, from (1), we have

$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x\big(t - h_i(t)\big) \\
&\quad + \sum_{i=m+1}^{p} A_i x\big(t - h_i(t)\big) \\
&\quad + f\Big(t, x(t), x\big(t - h_1(t)\big), \dots, x\big(t - h_p(t)\big)\Big) \\
&= A_0 x(t) + \sum_{i=1}^{m} A_i x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i(t)}^{t} \dot{x}(s)\, ds \\
&\quad + \sum_{i=m+1}^{p} A_i x\big(t - h_i(t)\big) \\
&\quad + f\Big(t, x(t), x\big(t - h_1(t)\big), \dots, x\big(t - h_p(t)\big)\Big) \\
&= \sum_{i=0}^{m} A_i x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i(t)}^{t} \Bigg[A_0 x(s) + \sum_{j=1}^{m} A_i x\big(s - h_j(s)\big) \\
&\qquad + \sum_{j=m+1}^{p} A_j x\big(s - h_j(s)\big) \\
&\qquad + f\Big(t, x(t), x\big(t - h_1(t)\big), \dots, x\big(t - h_p(t)\big)\Big) \Bigg] ds \\
&\quad + \sum_{i=m+1}^{p} A_i x\big(t - h_i(t)\big) \\
&\quad + f\Big(t, x(t), x\big(t - h_1(t)\big), \dots, x\big(t - h_p(t)\big)\Big) \\
&= Ax(t) - \sum_{i=1}^{m} A_i \int_{t-h_i(t)}^{t} \Bigg[A_0 x(s) + \sum_{j=1}^{m} A_i x\big(s - h_j(s)\big) \\
&\qquad + \sum_{j=m+1}^{p} A_j x\big(s - h_j(s)\big) \\
&\qquad + f\Big(t, x(t), x\big(t - h_1(t)\big), \dots, x\big(t - h_p(t)\big)\Big) \Bigg] ds \\
&\quad + \sum_{i=m+1}^{p} A_i x\big(t - h_i(t)\big) \\
&\quad + f\Big(t, x(t), x\big(t - h_1(t)\big), \dots, x\big(t - h_p(t)\big)\Big), \qquad t \geq 0, \qquad \text{(5a)}
\end{aligned}
$$

$$
x(t) = \theta(t), \qquad t \in [-H, 0]. \qquad \text{(5b)}
$$

Let $x(t) = [\exp(-\alpha t)]T^{-1}y(t)$, where T is any nonsingular matrix and $\alpha > 0$. Thus, from (5), we have

$$\begin{aligned}
\dot{y}(t) &= (TAT^{-1} + \alpha I)y(t) \\
&\quad -[\exp(\alpha t)] \cdot \sum_{i=1}^{m} A_i \int_{t-h_i(t)}^{t} \Bigg[[\exp(-\alpha s)] \cdot \left[TA_0T^{-1}\right] \cdot y(s) \\
&\qquad + \sum_{j=1}^{m} \left[\exp\left(-\alpha\left(s - h_j(s)\right)\right)\right] \cdot \left[TA_jT^{-1}\right] \cdot y\left(s - h_j(s)\right) \\
&\qquad + \sum_{j=m+1}^{p} \left[\exp\left(-\alpha\left(s - h_j(s)\right)\right)\right] \cdot \left[TA_jT^{-1}\right] \cdot y\left(s - h_j(s)\right) \\
&\qquad + g_1\left(s, y(s), y\left(s - h_1(s)\right), \ldots, y\left(s - h_p(s)\right)\right) \Bigg] ds \\
&\quad + \sum_{i=m+1}^{p} \left[\exp(\alpha \cdot h_i(t))\right] \cdot \left[TA_iT^{-1}\right] \cdot y\left(t - h_i(t)\right) \\
&\quad + g\left(t, y(t), y\left(t - h_1(t)\right), \ldots, y\left(t - h_p(t)\right)\right), \qquad \forall t \geq 0, \qquad (6a)
\end{aligned}$$

$$y(t) = [\exp(\alpha t)]Tx(t), \qquad t \in [-H, 0], \qquad (6b)$$

where

$$g_1(s, z_0, z_1, \ldots, z_p) = [\exp(-\alpha s)] \cdot g(s, z_0, z_1, \ldots, z_p)$$

and

$$\begin{aligned}
g\left(t, z_0, z_1, \ldots, z_p\right) &= [\exp(\alpha t)]T \cdot f\Big(t, [\exp(-\alpha t)]T^{-1}z_0, \\
&\quad \left[\exp\left(-\alpha\left(t - h_1(t)\right)\right)\right]T^{-1}z_1, \ldots, \left[\exp\left(-\alpha\left(t - h_p(t)\right)\right)\right]T^{-1}z_p\Big).
\end{aligned}$$

Define the dynamic system

$$\begin{aligned}
\dot{\bar{y}}(t) &= (TAT^{-1} + \alpha I)\bar{y}(t) \\
&\quad -[\exp(\alpha t)] \cdot \sum_{i=1}^{m} A_i \int_{t-h_i(t)}^{t} \Bigg[[\exp(-\alpha s)] \cdot \left[TA_0T^{-1}\right] \cdot \bar{y}(s) \\
&\qquad + \sum_{j=1}^{m} \left[\exp\left(-\alpha\left(s - h_j(s)\right)\right)\right] \cdot \left[TA_jT^{-1}\right] \cdot \bar{y}\left(s - h_j(s)\right) \\
&\qquad + \sum_{j=m+1}^{p} \left[\exp\left(-\alpha\left(s - h_j(s)\right)\right)\right] \cdot \left[TA_jT^{-1}\right] \cdot \bar{y}\left(s - h_j(s)\right) \\
&\qquad + g_1\left(s, \bar{y}(s), \bar{y}\left(s - h_1(s)\right), \ldots, \bar{y}\left(s - h_p(s)\right)\right) \Bigg] ds
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=m+1}^{p} \left[\exp(\alpha \cdot h_i(t))\right] \cdot \left[TA_iT^{-1}\right] \cdot \bar{y}(t - h_i(t)) \\
&+ g\Big(t, \bar{y}\big(t, \bar{y}(t - h_1(t)), \ldots, \bar{y}(t - h_p(t))\big)\Big), \qquad \forall t \geq 0, \\
&\coloneqq F_1(t, \bar{y}_t), \qquad \forall t \geq 0, \qquad (7a)
\end{aligned}
$$

$$\bar{y}(t) = y(t), \qquad t \in [-H, 0], \qquad (7b)$$

$$\bar{y}(t) = y(-H), \qquad t \in [-2H, -H], \qquad (7c)$$

where $\bar{y}_t$ is the segment of $\bar{y}(s)$ for $t - 2H \leq s \leq t$, $\bar{y}_t(s) \coloneqq y(t + s)$, $\forall s \in [-2H, 0]$, and $\|\bar{y}_t\|_s \coloneqq \sup_{-2H \leq r \leq 0} \|\bar{y}(t + r)\|$. By comparing (6) with (7), it is easy to see that $\bar{y}(t) = y(t), \forall t \geq 0$.

It is also easy to see that the function $D(x)$, defined in (4), is a strictly increasing function, $D(\infty) = \infty$, and $D(0) < 0$ in view of (2). Hence, there exists a unique $\bar{\alpha}$, with $\bar{\alpha} > 0$, such that $D(\bar{\alpha}) = 0$. Since $0 < \alpha < \bar{\alpha}$, we have

$$
\begin{aligned}
&\mu(TAT^{-1}) + \alpha + \left[\exp(2H\alpha)\right] \cdot H \cdot \left(\sum_{i=1}^{m} \|A_i\|\right) \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\|\right. \\
&\qquad \left. + \kappa(T) \cdot \sum_{i=0}^{p} a_i\right) + \left[\exp(H\alpha)\right] \cdot \left(\sum_{i=m+1}^{p} \|TA_iT^{-1}\|\right) \\
&\quad + \left[\exp(H\alpha)\right] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i = D(\alpha) < D(\bar{\alpha}) = 0. \qquad (8)
\end{aligned}
$$

This implies that there exists a sufficiently small positive constant ε_1 such that

$$
\begin{aligned}
k_1 \coloneqq\ &\mu(TAT^{-1}) + \alpha \\
&+ (1 + \varepsilon_1) \cdot \left[\exp(2H\alpha)\right] \cdot H \cdot \left(\sum_{i=1}^{m} \|A_i\|\right) \\
&\cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_i\right) \\
&+ (1 + \varepsilon_1) \cdot \left[\exp(H\alpha)\right] \cdot \left(\sum_{i=m+1}^{p} \|TA_iT^{-1}\|\right) \\
&+ (1 + \varepsilon_1) \cdot \left[\exp(H\alpha)\right] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i < 0. \qquad (9)
\end{aligned}
$$

By the definition of the function g and (A1), it can be deduced that

$$
\begin{aligned}
&\left\| g\big(t, \bar{y}(t), \bar{y}(t-h_1(t)), \ldots, \bar{y}\big(t-h_p(t)\big)\big) \right\| \\
&\quad = \Big\| [\exp(\alpha t)] \cdot T \cdot f\big(t, (\exp(-\alpha t)) T^{-1} \bar{y}(t), \\
&\qquad \times \big[\exp\big(-\alpha\big(t-h_1(t)\big)\big] T^{-1} \bar{y}\big(t-h_1(t)\big), \\
&\qquad \ldots, \big[\exp\big(-\alpha\big(t-h_p(t)\big)\big] T^{-1} \bar{y}\big(t-h_p(t)\big)\big) \Big\| \\
&\quad \leq [\exp(\alpha t)] \cdot \|T\| \cdot \sum_{i=0}^{p} a_i \cdot \Big\| \big[\exp\big(-\alpha\big(t-h_i(t)\big)\big] T^{-1} \bar{y}\big(t-h_i(t)\big) \Big\| \\
&\quad \leq [\exp(H\alpha)] \cdot \|T\| \cdot \sum_{i=0}^{p} a_i \cdot \|T^{-1}\| \cdot \|\bar{y}\big(t-h_i(t)\big)\| \\
&\quad = [\exp(H\alpha)] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i \cdot \|\bar{y}\big(t-h_i(t)\big)\| \\
&\quad \leq [\exp(H\alpha)] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i \cdot \|\bar{y}_t\|_s, \qquad \forall t \geq 0,
\end{aligned} \tag{10}
$$

with $h_0(t) = 0$. Furthermore, one has

$$
\begin{aligned}
&\Bigg\| -[\exp(\alpha t)] \cdot \sum_{i=1}^{m} A_i \int_{t-h_i(t)}^{t} \Bigg[[\exp(-\alpha s)] \cdot \big[T A_0 T^{-1}\big] \cdot \bar{y}(s) \\
&\qquad + \sum_{j=1}^{m} \big[\exp\big(-\alpha\big(s-h_j(s)\big)\big)\big] \cdot \big[T A_j T^{-1}\big] \cdot \bar{y}\big(s-h_j(s)\big) \\
&\qquad + \sum_{j=m+1}^{p} \big[\exp\big(-\alpha\big(s-h_j(s)\big)\big)\big] \cdot \big[T A_j T^{-1}\big] \cdot \bar{y}\big(s-h_j(s)\big) \\
&\qquad + g_1\big(s, \bar{y}(s), \bar{y}\big(s-h_1(s)\big), \ldots, \bar{y}\big(s-h_p(s)\big)\big) \Bigg] ds \Bigg\| \\
&\quad \leq [\exp(2H\alpha)] \cdot H \cdot \left(\sum_{i=1}^{m} \|A_i\| \right) \cdot \\
&\qquad \left(\sum_{i=0}^{p} \|T A_i T^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_i \right) \cdot \|\bar{y}_t\|_s, \qquad \forall t \geq 0,
\end{aligned} \tag{11}
$$

and

$$\left\| \sum_{i=m+1}^{p} \left[\exp\left(\alpha \cdot h_i(t)\right)\right] \cdot \left[TA_iT^{-1}\right] \cdot y\left(t - h_i(t)\right) \right\|$$
$$\leq \left[\exp(H\alpha)\right] \cdot \left(\sum_{i=m+1}^{p} \|TA_iT^{-1}\| \right) \cdot \|\bar{y}_t\|_s, \qquad \forall t \geq 0. \quad (12)$$

Hence the functional $F_1 : \Re \times \overline{C}(0) \to \Re^n$ takes $\Re \times$ (bounded sets of $\overline{C}(0)$) into bounded sets of $\Re^n$ in view of (10)–(12).

Let

$$B = TAT^{-1} + \alpha I$$

and

$$V(\bar{y}(t)) = \bar{y}^T(t)\bar{y}(t). \quad (13)$$

Thus, by Lemma 2 (ii), one has

$$\mu(B) = \mu(TAT^{-1}) + \alpha. \quad (14)$$

The time derivative of $V(\bar{y}(t))$ along the trajectories of the system (7) is given by

$$\begin{aligned}
\dot{V}(\bar{y}(t)) = {} & \bar{y}^T(t)(B + B^T)\bar{y}(t) \\
& - 2\bar{y}^T(t) \cdot \left[\exp(\alpha t)\right] \\
& \cdot \sum_{i=1}^{m} A_i \int_{t-h_i(t)}^{t} \Bigg[\left[\exp(-\alpha s)\right] \cdot \left[TA_0T^{-1}\right] \cdot \bar{y}(s) \\
& + \sum_{j=1}^{m} \left[\exp\left(-\alpha\left(s - h_j(s)\right)\right)\right] \cdot \left[TA_jT^{-1}\right] \cdot \bar{y}\left(s - h_j(s)\right) \\
& + \sum_{j=m+1}^{p} \left[\exp\left(-\alpha\left(s - h_j(s)\right)\right)\right] \cdot \left[TA_jT^{-1}\right] \cdot \bar{y}\left(s - h_j(s)\right) \\
& + g_1\left(s, \bar{y}(s), \bar{y}(s - h_1(s)), \ldots, \bar{y}\left(s - h_p(s)\right)\right) \Bigg] ds \\
& + 2\bar{y}^T(t) \cdot \sum_{i=m+1}^{p} \left[\exp\left(\alpha \cdot h_i(t)\right)\right] \cdot \left[TA_iT^{-1}\right] \cdot \bar{y}(t - h_i(t)) \\
& + 2\bar{y}^T(t) \cdot g\left(t, \bar{y}(t), \bar{y}(t - h_1(t)), \ldots, \bar{y}\left(t - h_p(t)\right)\right). \quad (15)
\end{aligned}$$

Applying (10)–(12) to (15), it yields

$$
\begin{aligned}
\dot{V}(\bar{y}(t)) \le{} & 2\mu(B)\|\bar{y}(t)\|^2 \\
& + 2\|\bar{y}(t)\| \cdot [\exp(2H\alpha)] \cdot H \cdot \left(\sum_{i=1}^{m} \|A_i\| \right) \\
& \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_i \right) \cdot \|\bar{y}\|_s \\
& + 2\|\bar{y}(t)\| \cdot [\exp(H\alpha)] \cdot \left(\sum_{i=m+1}^{p} \|TA_iT^{-1}\| \right) \cdot \|\bar{y}_t\|_s \\
& + 2\|\bar{y}(t)\| \cdot [\exp(H\alpha)] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i \|\bar{y}_t\|_s \\
={} & 2\mu(TAT^{-1}) \cdot \|\bar{y}(t)\|^2 + 2\alpha \cdot \|\bar{y}(t)\|^2 \\
& + 2\|\bar{y}(t)\| \cdot [\exp(2H\alpha)] \cdot H \cdot \left(\sum_{i=1}^{m} \|A_i\| \right) \\
& \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_i \right) \cdot \|\bar{y}_t\|_s \\
& + 2\|\bar{y}(t)\| \cdot [\exp(H\alpha)] \cdot \left(\sum_{i=m+1}^{p} \|TA_iT^{-1}\| \right) \cdot \|\bar{y}_t\|_s \\
& + 2\|\bar{y}(t)\| \cdot [\exp(H\alpha)] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i \|\bar{y}_t\|_s, \qquad (16)
\end{aligned}
$$

in view of (14). In the spirit of Lemma 1, with $p(s) = (1 + \varepsilon_1)^2 \cdot s$, we suppose that

$$V(\bar{y}(t + r)) < (1 + \varepsilon_1)^2 \cdot V(\bar{y}(t)), \qquad \forall -2H \le r \le 0,$$

which implies that

$$\|\bar{y}(t + r)\| < (1 + \varepsilon_1) \cdot \|\bar{y}(t)\|, \qquad \forall -2H \le r \le 0,$$

or that

$$\|\bar{y}_t\|_s \le (1 + \varepsilon_1) \cdot \|\bar{y}(t)\|, \qquad \forall t \in \Re^+. \qquad (17)$$

Substituting (17) into (16), it can be shown that

$$\begin{aligned}
\dot{V}(\bar{y}(t)) &\le 2\mu(TAT^{-1}) \cdot \|\bar{y}(t)\|^2 + 2\alpha \cdot \|\bar{y}(t)\|^2 \\
&\quad + 2(1+\varepsilon_1) \cdot [\exp(2H\alpha)] \cdot H \cdot \left(\sum_{i=1}^{m} \|A_i\| \right) \\
&\quad \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_i \right) \cdot \|\bar{y}(t)\|^2 \\
&\quad + 2(1+\varepsilon_1) \cdot [\exp(H\alpha)] \cdot \left(\sum_{i=m+1}^{p} \|TA_iT^{-1}\| \right) \cdot \|\bar{y}(t)\|^2 \\
&\quad + 2(1+\varepsilon_1) \cdot [\exp(H\alpha)] \cdot \kappa(T) \cdot \sum_{i=0}^{p} a_i \|\bar{y}(t)\|^2 \\
&= 2k_1 \cdot \|\bar{y}(t)\|^2,
\end{aligned} \tag{18}$$

in view of (9). Thus, by Lemma 1 with (7), (13), and (18), we conclude that the system (7) and the system (6) are both globally asymptotically stable, i.e., the system (1) is exponentially stable with the convergence rate α. This completes our proof. ▮

Remark 1. Although the function $D(x)$ defined in (4) is transcendental, the calculation of the unique positive zero $\bar{\alpha}$ such that $D(\bar{\alpha}) = 0$ is very easy. It is easy to see that $D(-\mu(TAT^{-1})) \ge 0$. Consequently, any one-dimensional line search method, e.g., Newton's method, can be used to find $\bar{\alpha}$ in the interval $[0, -\mu(TAT^{-1})]$.

Remark 2. The special case of the system (1) with commensurate time delays, i.e., $h_i(t) = i \cdot \tau, \forall i \in \underline{p}$, for some $\tau > 0$, has been considered in [11]. The special case of the system (1) with constant time delays, i.e., $h_i(t) = c_i \in \Re^+$, $\forall i \in \underline{p}$, has been considered in [16].

Remark 3. If we consider only the retarded systems with multiple time-varying delays, which is a special case of the system (1),

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x(t - h_i(t)), \qquad t \ge 0, \tag{19a}$$

$$x(t) = \theta(t), \qquad t \in [-H, 0], \tag{19b}$$

where $x \in \Re^n$ is the state vector, $A_i \in \Re^{n \times n}$, $\forall i \in \bar{p}$, $h_i(t)'s$, $\forall i \in \underline{p}$, are arbitrary delay arguments with $0 \le h_i(t) \le H$ for some constant $\bar{H} \ge 0$, and $\theta(t)$ is a continuous vector-valued function. Simply by setting $L := \varnothing$ and $L := \underline{p}$, respectively, in Theorem 1, we may obtain that the system (19) is exponentially stable provided that there exists a nonsingular matrix T such that either of the following conditions is satisfied:

$$\text{(i)} \qquad \mu\left(TA_0T^{-1}\right) + \sum_{i=1}^{p} \|TA_iT^{-1}\| < 0;$$

$$\text{(ii)} \qquad \mu(TAT^{-1}) + H \cdot \left(\sum_{i=1}^{p} \|A_i\|\right) \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\|\right) < 0,$$

where $A = \sum_{i=0}^{p} A_i$.

In particular, if we consider the system (19) with commensurate time delays, i.e., $h_i(t) = i \cdot \tau$, $\forall i \in \underline{p}$, for some $\tau > 0$, this result is less conservative than Corollary 3 in [17].

Remark 4. Our main result, i.e., Theorem 1, has a simple application to the stabilizability problem of the retarded control system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x\left(t - h_i(t)\right) + Bu(t), \qquad t \geq 0, \tag{20a}$$

$$x(t) = \theta(t), \qquad t \in [-H, 0], \tag{20b}$$

where $x \in \Re^n$ is the state vector; $A_i \in \Re^{n \times n}$, $\forall i \in \bar{p}$, $h_i(t)'s$, $\forall i \in \underline{p}$, are arbitrary delay arguments with $0 \leq h_i(t) \leq H$ for some constant $\bar{H} \geq 0$; $u \in \Re^m$ is the control; and $\theta(t)$ is a continuous vector-valued function. We assume that $A_i \neq 0$ for some $i \in \underline{p}$. In this remark, we will show that the system (20) is exponentially stabilizable by a constant state feedback $u(t) = Kx(t)$ provided that

$$\text{(i)} \qquad \left(\sum_{i=0}^{p} A_i, B\right) \text{ is a stabilizable pair;}$$

$$\text{(ii)} \qquad H \text{ is sufficiently small.}$$

This can be proved as follows. By (i), there exists a constant matrix K such that $A := (\sum_{i=0}^{p} A_i) + BK$ is a Hurwitz matrix with $-2a :=$

$\max \operatorname{Re}[\lambda_i, (A)] < 0$. The system (20) with the control $u(t) = Kx(t)$ can be represented as

$$\dot{x}(t) = (A_0 + BK)x(t) + \sum_{i=1}^{p} A_i x(t - h_i(t)), \qquad t \geq 0,$$
$$x(t) = \theta(t), \qquad t \in [-H, 0].$$

By Lemma 3, there exists a nonsingular matrix T such that

$$\mu(TAT^{-1}) \leq \max_i \operatorname{Re}[\lambda_i(A)] + a = -2a + a = -a < 0. \quad (21)$$

Hence one has

$$\mu(TAT^{-1}) + H \cdot \left(\sum_{i=1}^{p} \|A_i\| \right) \cdot \left(\|T(A_0 + BK)T^{-1}\| + \sum_{i=1}^{p} \|TA_iT^{-1}\| \right) < 0,$$

in view of (21) and (ii). Consequently, the system (20) is exponentially stabilized by the constant state feedback $u(t) = Kx(t)$ in view of Remark 3.

Remark 5. It is noted from Remark 4 that even if (A_0, B) is not a stabilizable pair, the system (20) may still be exponentially stabilized by a constant state feedback.

Remark 6. Note that if $h_i(t) = 0$, $\forall i \in \underline{p}$, or if $A_i = 0$, $\forall i \in \underline{p}$, the assertion of Remark 4 is reduced to the well-known fact that a delay-free linear system can be stabilized by a constant state feedback provided that $(\sum_{i=0}^{p} A_i, B)$ is a stabilizable pair.

3. ILLUSTRATIVE EXAMPLE

EXAMPLE 1. Consider the following uncertain retarded system with multiple time-varying delays:

$$\begin{aligned} \dot{x}(t) &:= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \\ &= A_0 x(t) + A_1 x(t - h_1(t)) + A_2 x(t - h_2(t)) \\ &\quad + f(t, x(t), x(t - h_1(t)), x(t - h_2(t))); \end{aligned} \quad (22)$$

where

$$A_0 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -2.8 & -0.1 \\ 0.1 & -3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & -0.2 \end{bmatrix},$$

$$f(t, x(t), x(t - h_1(t)), x(t - h_2(t)))$$
$$= \begin{bmatrix} a \cdot x_2(t - h_1(t)) \cdot \sin[t \cdot x_2^2(t - h_2(t))] \\ +b \cdot x_2(t) \cdot \sin[x_1(t - h_1(t)) \cdot x_2(t - h_2(t))] \\ c \cdot x_1(t - h_2(t)) \cdot \cos[t \cdot x_1(t - h_2(t)) \cdot x_2(t - h_1(t))] \end{bmatrix}, \quad (23)$$

with

$$-0.1 \le a, b, c \le 0.1 \qquad \text{and}$$
$$0 \le h_j(t) \le \max_{i \in \underline{2},\, t \in \Re} h_i(t) \le H := 0.2, \forall j \in \underline{2}, t \in \Re.$$

Comparing (22) with (1), one has $p = 2$. Comparing (A1) with (23), it can be deduced that $a_0 = a_1 = a_2 = 0.1$. In addition, from (3), by setting $L := \{1, 2\}, T := I$, it can be obtained that

$$\mu(TAT^{-1}) = -3, \qquad \|TA_0T^{-1}\| = 0.1414, \qquad \|TA_1T^{-1}\| = 3.0017,$$
$$\|TA_2T^{-1}\| = 0.2562, \qquad \|TAT^{-1}\| = 3.1, \qquad \kappa(T) = 1.$$

This implies that

$$\mu(TAT^{-1}) + H \cdot \left(\sum_{i \in L} \|A_i\| \right) \cdot \left(\sum_{i=0}^{p} \|TA_iT^{-1}\| + \kappa(T) \cdot \sum_{i=0}^{p} a_1 \right)$$
$$+ \left(\sum_{i \in \underline{p} \setminus L} \|TA_iT^{-1}\| \right) + \kappa(T) \sum_{i=0}^{p} a_i \le -0.2896 < 0.$$

Furthermore, $\overline{\alpha} = 0.1413$ is the unique positive zero of (4). Hence, by Theorem 1, the system (22) is exponentially stable with the guaranteed convergence rate $\alpha = \overline{\alpha} - \varepsilon = 0.14$ by selecting $\varepsilon = 0.0013$.

With, e.g.,

$$a = b = c = 0.1, \qquad h_1(t) = 0.1 + 0.1 \sin(11t),$$
$$h_2(t) = 0.1 - 0.1 \cos(2t),$$

some typical phase trajectories of the system (22) are depicted in Fig. 1.

It is noted that, in the case of the system (22) with $h_1(t) = 0.1$ and $h_2(t) = 0.2$, the asymptotic stability of the system (22) cannot be guaranteed by Theorem 1 in [11], but the exponential stability of the system (22) can be guaranteed by our Theorem 1.

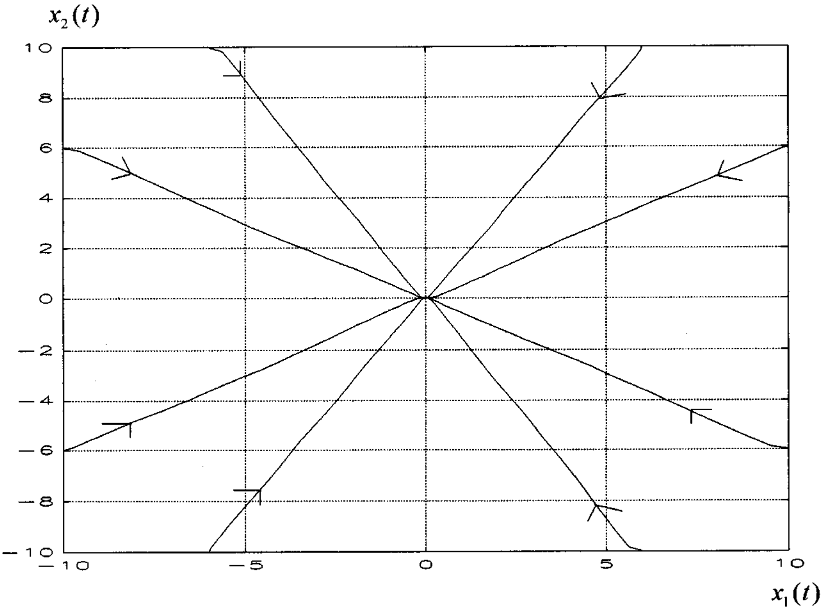

FIG. 1. Typical phase trajectories of the system (22).

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